Partial Fraction

Liming Pang

We are going to discuss about how to evaluate integrals of form $\int \frac{P(x)}{Q(x)}\, dx$, where $P(x)$ and $Q(x)$ are polynomials.

Theorem 1. *Every non-constant polynomial can be written as a product of some polynomials of degree one or two.*

Theorem 2. *If $P(x)$ and $Q(x)$ are polynomials, then we can write $\frac{P(x)}{Q(x)}$ as a sum of a polynomial and some partial fractions of the form $\frac{A}{(ax+b)^i}$ or $\frac{Ax+B}{(ax^2+bx+c)^j}$, and the denominator of each partial fraction is a factor of $Q(x)$.*

The above theorems can be proved by Algebra, and we omit the proof here.

So when dealing with $\int \frac{P(x)}{Q(x)}\, dx$, as indicated by the above two theorems, we can do the following steps:

1. If the degree of $P(x)$ is greater than or equal to the degree of $Q(x)$, we first write it in a form $\frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}$ with the degree of $R(x)$ less than the degree of $Q(x)$.
2. Factorise $Q(x)$ as a product of linear and quadratic polynomials.
3. Write $\frac{R(x)}{Q(x)}$ as a sum of partial fractions.
4. Evaluate the integral

Example 3. *Compute* $\int \frac{x^2+2x-1}{2x^3+3x^2-2x}\, dx$

$(2x^3 + 3x^2 - 2x) = x(2x^2 + 3x - 2) = x(2x + 1)(x - 2)$, *so we let*

$$\frac{x^2 + 2x - 1}{2x^3 + 3x^2 - 2x} = \frac{A}{x} + \frac{B}{2x + 1} + \frac{C}{x - 2}$$

This implies

$$x^2+2x-1 = A(2x-1)(x+2)+Bx(x+2)+Cx(2x-1) = (2A+B+2C)x^2+(3A+2B-C)x-2A$$

So we obtain equations:

$$\begin{cases} 1 = 2A + B + 2C \\ 2 = 3A + 2B - C \\ -1 = -2A \end{cases}$$

And solving the equations, we see $A = \frac{1}{2}$, $B = \frac{1}{5}$, $C = -\frac{1}{10}$

We can do the integration:

$$\begin{aligned} \int \frac{x^2+2x-1}{2x^3+3x^2-2x}\,dx &= \int \frac{1}{2x} + \frac{1}{5(2x-1)} - \frac{1}{10(x+2)}\,dx \\ &= \frac{\ln|x|}{2} + \frac{\ln|2x-1|}{10} - \frac{\ln|x+2|}{10} + C \end{aligned}$$

Example 4. *Compute* $\int \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}\,dx$

We first notice the degree of the numerator is no less than that of the denominator, so we do a polynomial division to write

$$\frac{x^4-2x^2+4x+1}{x^3-x^2-x+1} = (x+1) + \frac{4x}{x^3-x^2-x+1}$$

The factorisation of the denominator is

$$x^3-x^2-x+1 = x^2(x-1)-(x-1) = (x^2-1)(x-1) = (x+1)(x-1)^2$$

Now let

$$\begin{aligned} \frac{4x}{(x+1)(x-1)^2} &= \frac{A}{x+1} + \frac{B}{(x-1)} + \frac{C}{(x-1)^2} \\ &= \frac{A(x-1)^2 + B(x+1)(x-1) + C(x+1)}{(x-1)(x+1)^2} \\ &= \frac{(A+B)x^2 + (C-2A)x + (A-B+C)}{(x+1)(x-1)^2} \end{aligned}$$

We get

$$\begin{cases} A + B = 0 \\ C - 2A = 4 \\ A - B + C = 0 \end{cases}$$

The solution is $A = -1, B = 1, C = 2$, so

$$\frac{4x}{(x+1)(x-1)^2} = -\frac{1}{x+1} - \frac{1}{(x-1)} + \frac{2}{(x-1)^2}$$

$$\begin{aligned} \int \frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1}\, dx &= \int (x+1) - \frac{1}{x+1} - \frac{1}{(x-1)} + \frac{2}{(x-1)^2}\, dx \\ &= \frac{1}{2}(x+1)^2 - \ln|x+1| + \ln|x-1| - \frac{2}{x-1} + C \end{aligned}$$

Example 5. *Compute* $\int \frac{2x^2 - x + 4}{x^3 + 4x}\, dx$

The denominator is $x^3 + 4x = x(x^2 + 4)$

Let

$$\begin{aligned} \frac{2x^2 - x + 4}{x(x^2+4)} &= \frac{A}{x} + \frac{Bx + C}{x^2 + 4} \\ &= \frac{A(x^2+4) + (Bx + C)x}{x(x^2+4)} \\ &= \frac{(A+B)x^2 + Cx + 4A}{x(x^2+4)} \end{aligned}$$

We get

$$\begin{cases} A + B = 2 \\ C = -1 \\ 4A = 4 \end{cases}$$

The solution is $A = 1, B = 1, C = -1$

This means

$$\frac{2x^2 - x + 4}{x^3 + 4x} = \frac{1}{x} + \frac{x}{x^2+4} - \frac{1}{x^2+4}$$

$$\begin{aligned} \int \frac{2x^2 - x + 4}{x^3 + 4x}\, dx &= \int \frac{1}{x} + \frac{x}{x^2+4} - \frac{1}{x^2+4}\, dx \\ &= \ln|x| + \frac{1}{2}\ln(x^2+4) - \frac{1}{2}\tan^{-1}(\frac{x}{2}) + C \end{aligned}$$

Example 6. *Compute* $\int \frac{1-x+2x^2-x^3}{x(x^2+1)^2}\,dx$

Let

$$\begin{aligned}
\frac{1-x+2x^2-x^3}{x(x^2+1)^2} &= \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2} \\
&= \frac{A(x^2+1)^2 + (Bx+C)x(x^2+1) + (Dx+E)x}{x(x^2+1)^2} \\
&= \frac{(A+B)x^4 + Cx^3 + (2A+B+D)x^2 + (C+E)x + A}{x(x^2+1)^2}
\end{aligned}$$

We get

$$\begin{cases} A+B=0 \\ C=-1 \\ 2A+B+D=2 \\ C+E=-1 \\ A=1 \end{cases}$$

The solution is $A=1, B=-1, C=-1, D=1, E=0$.
So

$$\frac{1-x+2x^2-x^3}{x(x^2+1)^2} = \frac{1}{x} - \frac{x}{x^2+1} - \frac{1}{x^2+1} + \frac{x}{(x^2+1)^2}$$

$$\begin{aligned}
\int \frac{1-x+2x^2-x^3}{x(x^2+1)^2}\,dx &= \int \frac{1}{x} - \frac{x}{x^2+1} - \frac{1}{x^2+1} + \frac{x}{(x^2+1)^2}\,dx \\
&= \ln|x| - \frac{1}{2}\ln(x^2+1) - \tan^{-1}(x) - \frac{1}{2(x^2+1)} + C
\end{aligned}$$